Distance Computation for Quadratic Complexes

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Conics, Quadrics and Quadratic Complexes

- Quadratic Complexes are polyhedra with faces embedded on quadrics and conics as edges.
- A quadric is given by an algebraic equation of degree 2:

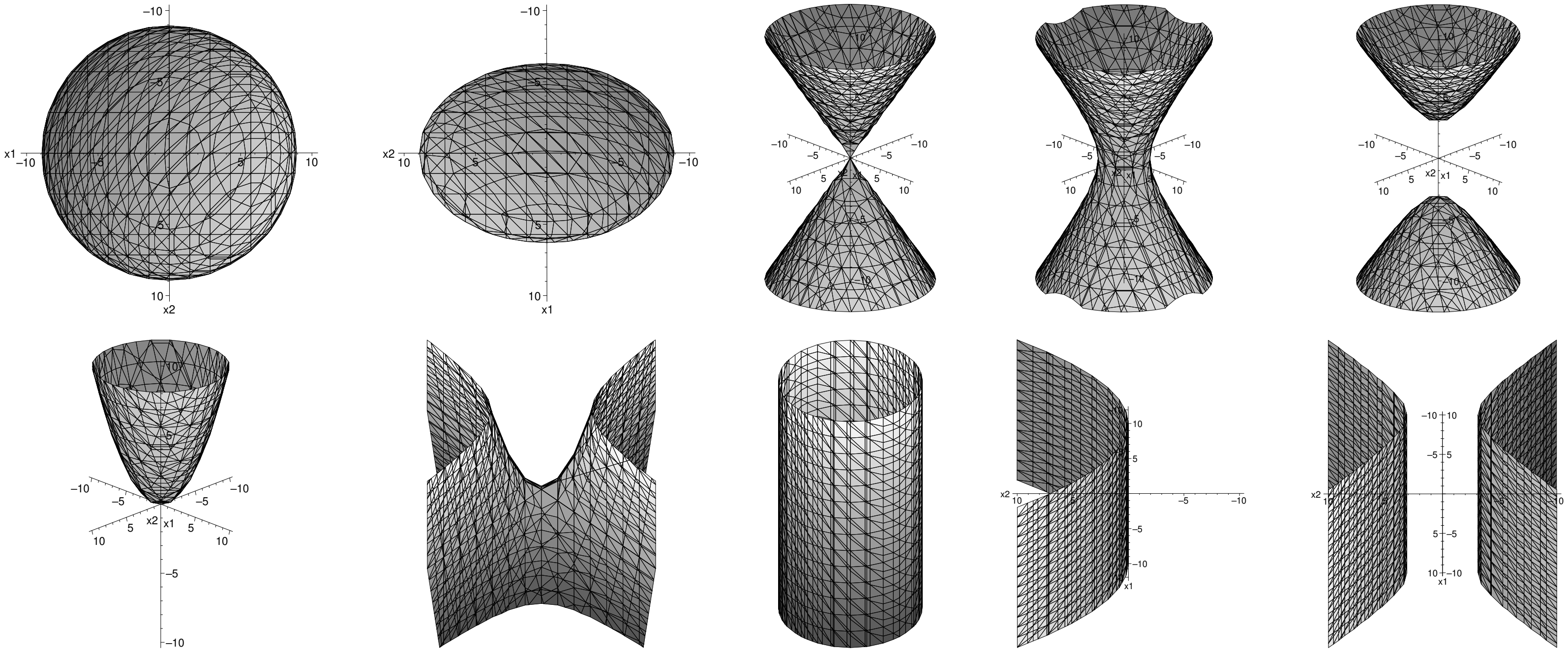
$$\{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{a}^T\boldsymbol{x} + a_0 = 0\},$$

  for a vector $\boldsymbol{a} \in \mathbb{R}^3$ and symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$.
- A conic is explicitly given as the following point set:

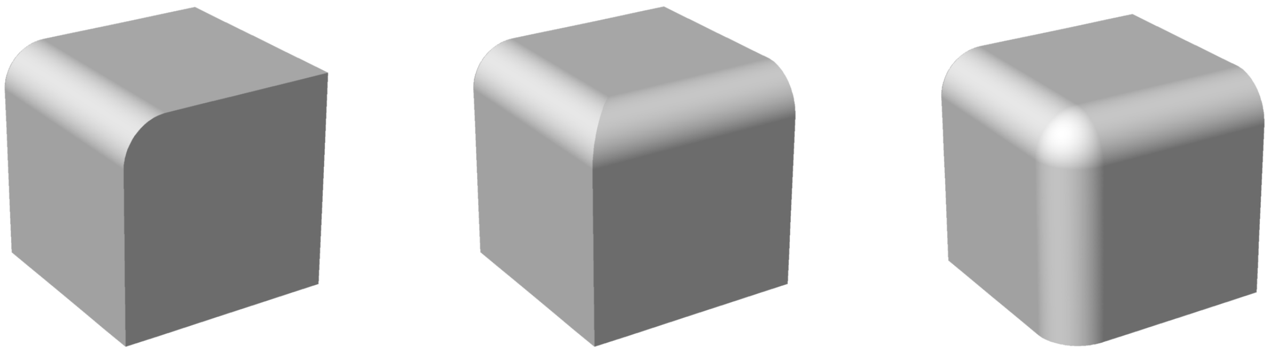
$$\{\boldsymbol{p} \in \mathbb{R}^3 \mid \boldsymbol{p} \;=\; \boldsymbol{c} + r(t)\boldsymbol{u} + s(t)\boldsymbol{v}\},$$

  where $(r, s) \in \{(\sin, \cos), (\sinh, \cosh), (\mathrm{id}, 0), (\mathrm{id}, \mathrm{id}^2)\}$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ with $\boldsymbol{u}^T\boldsymbol{v} = 0$.

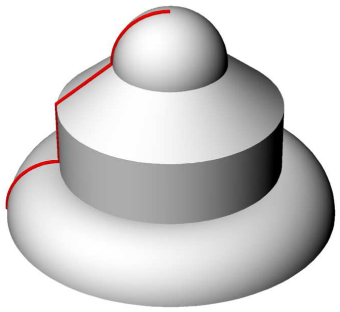
Examples of Quadrics



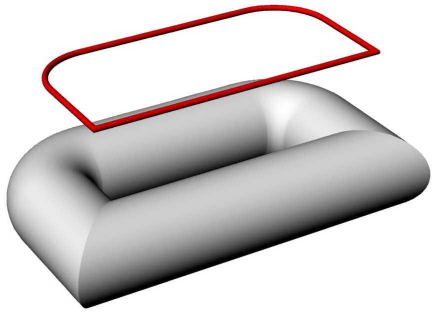
Quadratic Complexes in CAD I



Filleting

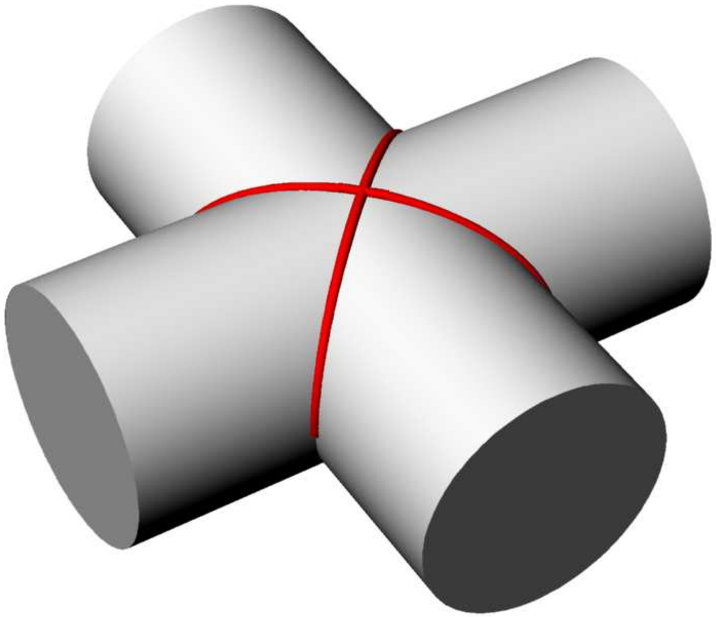


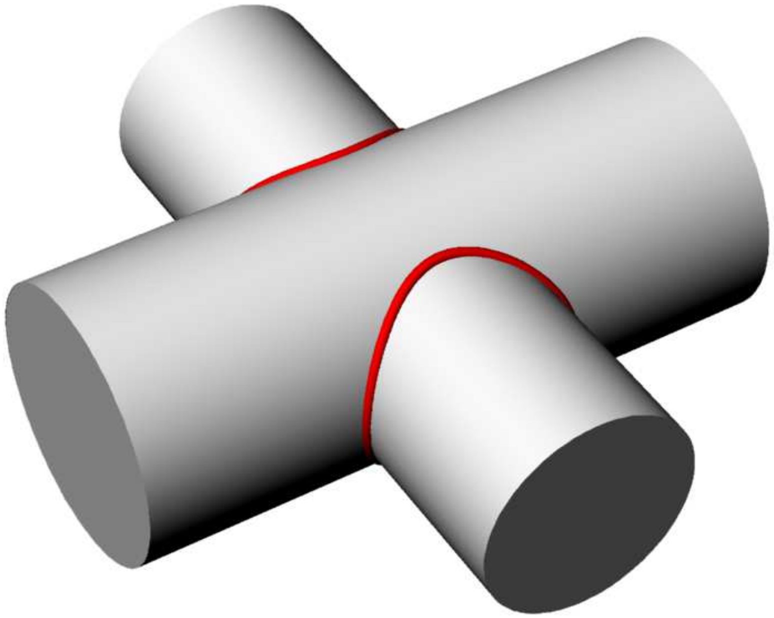
Revolving



Tubing

Quadratic Complexes in CAD II





Boolean Operations (Union)

Normal Forms of Quadrics

Central Surfaces: det($\boldsymbol{A}$) $\neq 0$		
Ellipsoids / Hyperboloids	$\boldsymbol{a} = \mathbf{0}$	$a_0 \neq 0$
Cone	$\boldsymbol{a} = \mathbf{0}$	$a_0 = 0$

Non-Central Surfaces: det($\boldsymbol{A}$) $= 0$			
Paraboloids	$A_3 = 0$	$a_3 \neq 0$	$a_0 = 0$
Elliptical /Hyperbolical Cylinder	$A_3 = 0$	$\boldsymbol{a} = \mathbf{0}$	$a_0 \neq 0$
Parabolical Cylinder	$A_1 = A_3 = 0$	$a_1 \neq 0$	$a_0 = 0$

The Distance Computation Problem

Definition 1. *Given two quadratic complexes $\boldsymbol{C}_1$, $\boldsymbol{C}_2$. The distance computation problem is to determine the global minimum of the distance function δ between the respective point sets, together with a pair of witness points i.e.*

(i) the value $\delta^* := \delta(\boldsymbol{C}_1, \boldsymbol{C}_2)$,

(ii) a pair of points $(\boldsymbol{p}, \boldsymbol{q})$, s.t. $\delta^* = \delta(\boldsymbol{p}, \boldsymbol{q})$,

where δ denotes the EUCLIDEAN distance function between two points or set of points respectively.

Closest Points Between Faces

Let F_1 and F_2 be disjoint faces of Quadratic Complexes that are embedded on the quadratic surfaces Q_1 and Q_2, where

$$\begin{aligned} Q_1 &:= \{\boldsymbol{x} \mid \boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{a}^T \boldsymbol{x} + a_0 = 0\}, \\ Q_2 &:= \{\boldsymbol{y} \mid \boldsymbol{y}^T \boldsymbol{B}\boldsymbol{y} + 2\boldsymbol{b}^T \boldsymbol{y} + b_0 = 0\}. \end{aligned}$$

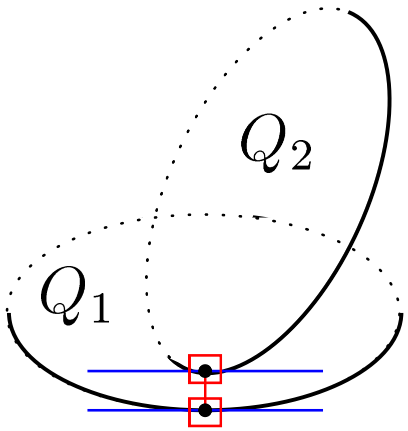
If $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ is a pair of closest points between F_1 and F_2, then either

(i) $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ is an extremum of the quadratic distance function between Q_1 and Q_2 i.e. there are $\alpha, \beta \in \mathbb{R},\ \alpha, \beta \neq 0$ s.t.

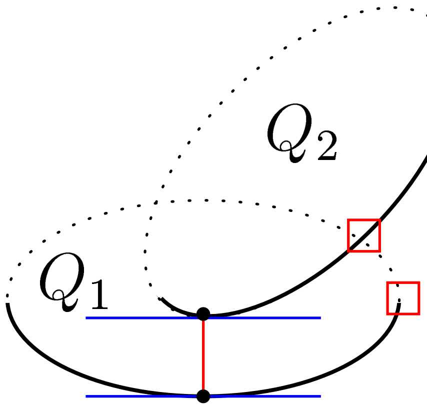
$$\boldsymbol{n}(\boldsymbol{p}_1) = \alpha(\boldsymbol{p}_2 - \boldsymbol{p}_1) \qquad \boldsymbol{n}(\boldsymbol{p}_2) = \beta(\boldsymbol{p}_1 - \boldsymbol{p}_2),$$

where $\boldsymbol{n}(\boldsymbol{p}_i)$ denotes the normal of Q_i in $\boldsymbol{p}_i$, or

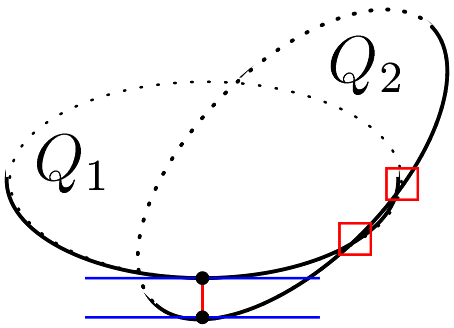
(ii) $\boldsymbol{p}_1$, or $\boldsymbol{p}_2$ lies on the boundary of the face F_1 or F_2, respectively.



$f_1 \cap f_2 = \emptyset$: case (i).



$f_1 \cap f_2 \neq \emptyset$: case (ii).



$f_1 \cap f_2 \neq \emptyset$: Precondition violated.

A Generic Algorithm

Input: Entities E_1 and E_2 of type face, edge or vertex.
Output: $\delta(E_1, E_2)$ and a pair of closest points $(\boldsymbol{p}_1, \boldsymbol{p}_2)$.

ENTITYDISTANCE(E_1, E_2)
    $[isDisjoint, (\boldsymbol{p}_1, \boldsymbol{p}_2)] \leftarrow$ INTERSECT(E_1, E_2)
    if $isDisjoint = false$
        return $[0, (\boldsymbol{p}_1, \boldsymbol{p}_2)]$
    $\delta_G \leftarrow \infty$
    while $[\delta, (\boldsymbol{q}_1, \boldsymbol{q}_2)] \leftarrow$ EXTREMA(E_1, E_2)
        if $(\boldsymbol{q}_1 \in E_1)$ **and** $(\boldsymbol{q}_2 \in E_2)$
            if $\delta < \delta_G$
                $\delta_G \leftarrow \delta, \quad (\boldsymbol{p}_1, \boldsymbol{p}_2) \leftarrow (\boldsymbol{q}_1, \boldsymbol{q}_2)$
    if E_1 is not a vertex
        foreach subentity E **of** E_1
            $[\delta, (\boldsymbol{q}_1, \boldsymbol{q}_2)] \leftarrow$ ENTITYDISTANCE(E, E_2)
            if $\delta < \delta_G$
                $\delta_G \leftarrow \delta, \quad (\boldsymbol{p}_1, \boldsymbol{p}_2) \leftarrow (\boldsymbol{q}_1, \boldsymbol{q}_2)$
    if E_2 is not a vertex
        foreach subentity E **of** E_2
            $[\delta, (\boldsymbol{q}_1, \boldsymbol{q}_2)] \leftarrow$ ENTITYDISTANCE(E_1, E)
            if $\delta < \delta_G$
                $\delta_G \leftarrow \delta, \quad (\boldsymbol{p}_1, \boldsymbol{p}_2) \leftarrow (\boldsymbol{q}_1, \boldsymbol{q}_2)$
    return $[\delta_G, (\boldsymbol{p}_1, \boldsymbol{p}_2)]$

Main Result

Theorem 1. *The distance between two faces of quadratic complexes can be computed by solving systems of univariate and bivariate polynomials in which the degree of every variable is at most 6. These systems can be solved by finding the roots of univariate polynomials of degree at most 24.*

Our Approach

- Point–Curve
  Point–Surface
  Curve–Curve
  Curve–Surface
  Surface–Surface
- ***Lagrange–formalism:***
  Deriving univariate / bivariate systems
- ***Elimination Theory:***
  Reducing to univariate Resultant polynomial
- ***Geometrical, Algebraical Insight:***
  Factorization
- **Univariate Polynomials to solve**

The Point-Surface Case

The LAGRANGE formalism for the point-surface problem, gives

$$\begin{aligned}
\mathcal{L}(\boldsymbol{x};\alpha) &= (\boldsymbol{x}-\boldsymbol{p})^2 + \alpha(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{a}^T\boldsymbol{x} + a_0), \\
\frac{\partial\mathcal{L}(.)}{\partial\boldsymbol{x}} = 0 &\iff \alpha(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{a}) = \boldsymbol{p}-\boldsymbol{x}, \\
\frac{\partial\mathcal{L}(.)}{\partial\alpha} = 0 &\iff \boldsymbol{x}^T A\boldsymbol{x} + 2\boldsymbol{a}^T\boldsymbol{x} + a_0 = 0.
\end{aligned}$$

From the first LAGRANGE-condition, we can derive:

$$\boldsymbol{x} \;=\; (\boldsymbol{E}+\alpha\boldsymbol{A})^{-1}(\boldsymbol{p}-\alpha\boldsymbol{a}) \;=:\; \boldsymbol{D}_\alpha^{-1}\boldsymbol{p}_\alpha.$$

Substituting $\boldsymbol{x}$ in the second equation gives the univariate system:

$$f(\alpha) \;=\; \boldsymbol{p}_\alpha^T\overline{\boldsymbol{D}}_\alpha\boldsymbol{A}\overline{\boldsymbol{D}}_\alpha\boldsymbol{p}_\alpha + 2\boldsymbol{a}^T\overline{\boldsymbol{D}}_\alpha\boldsymbol{p}_\alpha\boldsymbol{a}|\boldsymbol{D}_\alpha| + a_0|\boldsymbol{D}_\alpha|^2 \;=\; 0.$$

Examples

$$f(\alpha) = A_1 p_{\alpha 1}^2 d_2^2 d_3^2 + A_2 p_{\alpha 2}^2 d_1^2 d_3^2 + A_3 p_{\alpha 3}^2 d_1^2 d_2^2 + a_0 d_1^2 d_2^2 d_3^2 + 2(a_1 p_{\alpha 1} d_1 d_2^2 d_3^2 + a_2 p_{\alpha 2} d_1^2 d_2 d_3^2 + a_3 p_{\alpha 3} d_1^2 d_2^2 d_3) = 0$$

Central Surfaces:

Ellipsoid / Hyperboloid: $\boldsymbol{a} = \mathbf{0} \;\Rightarrow\; \boldsymbol{p}_\alpha = \boldsymbol{p}$

$$f(\alpha) = A_1 p_1^2 d_2^2 d_3^2 + A_2 p_2^2 d_1^2 d_3^2 + A_3 p_3^2 d_1^2 d_2^2 + a_0 d_1^2 d_2^2 d_3^2 = 0$$

Non-Central Surfaces:

Paraboloids:

$A_3 = 0,\ a_1 = a_2 = 0,\ a_0 = 0 \;\Rightarrow\; d_3 = 1,\ p_{\alpha 1} = p_1,\ p_{\alpha 2} = p_2$

$$f(\alpha) = A_1 p_1^2 d_2^2 + A_2 p_2^2 d_1^2 + 2 a_3 p_{\alpha 3} d_1^2 d_2^2 = 0$$

Summary: Point-Surface-Case

Point - Central Surface		
Ellipsoid	Hyperboloid	Cone
6	6	4

Point - Non-Central Surface		
Paraboloids	Elliptical / Hyperbolical Cylinders	Parabolical Cylinder
5	4	3

The Curve-Surface Case

If we substitute $\boldsymbol{p}$ by the explicit representation of a conic, i.e.

$$P: \quad \boldsymbol{p}(t) \ = \ c + r(t)\boldsymbol{u} + s(t)\boldsymbol{v}.$$

then we get a third LAGRANGE-condition

$$\frac{\partial \mathcal{L}(.)}{\partial t} \ = \ 0 \quad \Longleftrightarrow \quad (\boldsymbol{x} - \boldsymbol{p})^T \frac{\partial \boldsymbol{p}}{\partial t} \ = \ 0.$$

and in contrast to the point-surface case a bivariate system of equations:

$$\begin{aligned} f(\alpha, t) &= \boldsymbol{p}_\alpha^T \overline{\boldsymbol{D}}_\alpha \boldsymbol{A} \overline{\boldsymbol{D}}_\alpha \boldsymbol{p}_\alpha + 2\boldsymbol{a}^T \overline{\boldsymbol{D}}_\alpha \boldsymbol{p}_\alpha \boldsymbol{a} |\boldsymbol{D}_\alpha| + a_0 |\boldsymbol{D}_\alpha|^2 \ = \ 0, \\ g(\alpha, t) &= \left(\overline{\boldsymbol{D}}_\alpha \boldsymbol{p}_\alpha - |\boldsymbol{D}_\alpha| \boldsymbol{p}\right) \frac{\partial \boldsymbol{p}}{\partial t} \ = \ 0. \end{aligned}$$

Example: Central Surfaces

$$
\begin{aligned}
f(\alpha,t) &= A_1p_1^2d_2^2d_3^2 + A_2p_2^2d_1^2d_3^2 + A_3p_3^2d_1^2d_2^2 + a_0d_1^2d_2^2d_3^2 &= 0,\\
g(\alpha,t) &= A_1p_1p_1'd_2d_3 + A_2p_2p_2'd_1d_3 + A_3p_3p_3'd_1d_2 &= 0.
\end{aligned}
$$

	Ellipse	**Hyperbola**	**Parabola**	**Line**
$r(t),\ s(t)$	$\frac{1-t^2}{1+t^2}$ $\frac{2t}{1+t^2}$	$\frac{1+t^2}{1-t^2}$ $\frac{2t}{1-t^2}$	t t^2	t 0
$deg(f,\alpha)$	6	6	6	6
$deg(f,t)$	4	4	4	2
$deg(f,\alpha,t)$	10	10	10	8
$deg(g,\alpha)$	2	2	2	2
$deg(g,t)$	4	4	3	1
$deg(g,\alpha,t)$	6	6	5	3

Factorization of the Resultant Polynomial I

Lemma 1. *Let $f = g = 0$ be our system of equations, i.e.*

$$\begin{aligned} f(\alpha, t) &= A_1 p_1^2 d_2^2 d_3^2 + A_2 p_2^2 d_1^2 d_3^2 + A_3 p_3^2 d_1^2 d_2^2 + a_0 d_1^2 d_2^2 d_3^2 &= 0, \\ g(\alpha, t) &= A_1 p_1 p_1' d_2 d_3 + A_2 p_2 p_2' d_1 d_3 + A_3 p_3 p_3' d_1 d_2 &= 0. \end{aligned}$$

and let α_i denote the root of d_i, $i = 1, 2, 3$. Then

(i) The pair (α_i, t_i) is a solution of the bivariate system for every t_i solving the equation $p_i = 0$, $i = 1, 2, 3$,

(ii) If the curve is not a line, every α_i is a root of multiplicity 4 in $Res(f, g, t)$ whereas every t_i has multiplicity 2 in $Res(f, g, \alpha)$.

Factorization of the Resultant Polynomial II

Corollary 1. *If the curve is not a line, the Resultant Polynomial can be written as the following product:*

$$\begin{aligned} Res(f,g,t) &= h_\alpha \prod_{i=1}^{3} d_i^4 = h_\alpha \prod_{i=1}^{3} (\alpha - \alpha_i)^4, \\ Res(f,g,\alpha) &= h_t \prod_{i=1}^{3} p_i^2 = h_t \prod_{i=1}^{3} (t - t_{i1})^2 (t - t_{i2})^2. \end{aligned}$$

where h_α and h_t are univariate polynomials of degree at most 20.

Summary: Curve - Central-Surface Case

	Ellipsoid	Hyperboloids	Cone
Ellipse	20	20	12
Hyperbola	20	20	12
Parabola	14	14	8
Line	4	4	2

Summary: Curve - Non-Central-Surface Case

	Paraboloids	Elliptical / Hyperbolical Cylinders	Parabolical Cylinder
Ellipse	16	12	8
Hyperbola	16	12	8
Parabola	11	8	5
Line	3	2	1

The Surface-Surface Case

By setting up the LAGRANGE formalism for the problem

$$\min\ (\boldsymbol{x}-\boldsymbol{y})^2, \quad \boldsymbol{x} \in Q_1, \boldsymbol{y} \in Q_2$$

we get the LAGRANGE function $\mathcal{L}$ and -conditions $(i), \ldots, (iv)$:

$$\begin{aligned} \mathcal{L}(\boldsymbol{x},\boldsymbol{y};\alpha,\beta) \quad = \quad & (\boldsymbol{x}-\boldsymbol{y})^2 + \alpha(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{a}^T\boldsymbol{x} + a_0) \\ & + \beta(\boldsymbol{y}^T\boldsymbol{B}\boldsymbol{y} + 2\boldsymbol{b}^T\boldsymbol{y} + b_0) \end{aligned}$$

$$\begin{aligned}
(i) \quad & \partial\frac{\mathcal{L}(.)}{\partial \boldsymbol{x}} = 0 & \Longleftrightarrow \quad & \alpha(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{a}) = \boldsymbol{y}-\boldsymbol{x} \\
(ii) \quad & \partial\frac{\mathcal{L}(.)}{\partial \boldsymbol{y}} = 0 & \Longleftrightarrow \quad & \beta(\boldsymbol{B}\boldsymbol{y}+\boldsymbol{b}) = \boldsymbol{x}-\boldsymbol{y} \\
(iii) \quad & \partial\frac{\mathcal{L}(.)}{\partial \alpha} = 0 & \Longleftrightarrow \quad & \boldsymbol{x}^T A\boldsymbol{x} + 2\boldsymbol{a}^T\boldsymbol{x} + a_0 = 0 \\
(iv) \quad & \partial\frac{\mathcal{L}(.)}{\partial \beta} = 0 & \Longleftrightarrow \quad & \boldsymbol{y}^T B\boldsymbol{y} + 2\boldsymbol{b}^T\boldsymbol{y} + b_0 = 0
\end{aligned}$$

Solving The Lagrange System

By setting $\lambda := 1/\alpha$ and $\mu := 1/\beta$ we can derive from (i) and (ii):

$$\begin{aligned} \boldsymbol{x} &= -(\boldsymbol{BA} + \lambda\boldsymbol{B} + \mu\boldsymbol{A})^{-1}(\boldsymbol{Ba} + \lambda\boldsymbol{b} + \mu\boldsymbol{a}) \;=:\; -\frac{\overline{\boldsymbol{C}}_{\lambda,\mu}}{|\boldsymbol{C}_{\lambda,\mu}|}\boldsymbol{c}_B, \\ \boldsymbol{y} &= -(\boldsymbol{AB} + \lambda\boldsymbol{B} + \mu\boldsymbol{A})^{-1}(\boldsymbol{Ab} + \lambda\boldsymbol{b} + \mu\boldsymbol{a}) \;=:\; -\frac{\overline{\boldsymbol{C}}^T_{\lambda,\mu}}{|\boldsymbol{C}_{\lambda,\mu}|}\boldsymbol{c}_A, \end{aligned}$$

where $\overline{\boldsymbol{C}}_{\lambda,\mu}$ denotes the adjoint and $|\boldsymbol{C}_{\lambda,\mu}|$ the determinant of $\boldsymbol{C}_{\lambda,\mu}$.

Substituting $\boldsymbol{x}$ and $\boldsymbol{y}$ in (iii) and (iv) we get the system:

$$\begin{aligned} f(\lambda,\mu) &= \boldsymbol{c}_B^T\overline{\boldsymbol{C}}^T_{\lambda,\mu}\boldsymbol{A}\overline{\boldsymbol{C}}_{\lambda,\mu}\boldsymbol{c}_B - 2|\boldsymbol{C}_{\lambda,\mu}|\boldsymbol{a}^T\overline{\boldsymbol{C}}_{\lambda,\mu}\boldsymbol{c}_B + a_0|\boldsymbol{C}_{\lambda,\mu}|^2 = 0, \\ g(\lambda,\mu) &= \boldsymbol{c}_A^T\overline{\boldsymbol{C}}_{\lambda,\mu}\boldsymbol{B}\overline{\boldsymbol{C}}^T_{\lambda,\mu}\boldsymbol{c}_A - 2|\boldsymbol{C}_{\lambda,\mu}|\boldsymbol{b}^T\overline{\boldsymbol{C}}^T_{\lambda,\mu}\boldsymbol{c}_A + b_0|\boldsymbol{C}_{\lambda,\mu}|^2 = 0, \end{aligned}$$

The Inverse of $C_{\lambda,\mu}$

Proposition 1. *The adjoint and determinant of $C_{\lambda,\mu} = BA + \lambda B + \mu A$ is given by*

$$\begin{aligned}
\overline{C_{\lambda,\mu}} &= \overline{B}\lambda^2 + \overline{A}\mu^2 + T_A\overline{B}\lambda + \overline{A}T_B\mu + (T_BT_A - T_{AB})\lambda\mu + \overline{A}\,\overline{B}, \\
|C_{\lambda,\mu}| &= |B|\lambda^3 + |A|\mu^3 + |B|\mathit{tr}(A)\lambda^2 + |A|\mathit{tr}(B)\mu^2 + \\
&\quad |B|\mathit{tr}(\overline{A})\lambda + |A|\mathit{tr}(\overline{B})\mu + \mathit{tr}(\overline{B}A)\lambda^2\mu + \mathit{tr}(\overline{A}B)\lambda\mu^2 + \\
&\quad \big(\mathit{tr}(\overline{A})\mathit{tr}(\overline{B}) - \mathit{tr}(\overline{A}\,\overline{B})\big)\lambda\mu + |A||B|,
\end{aligned}$$

where $T_M := \mathit{tr}(M)E - M$ for a matrix $M \in \mathbb{R}^{3\times 3}$.

Corollary 2. *The polynomials f and g have degree 6 in λ as well as μ. Moreover the total degree of f and g is also 6.*

Corollary 3. *(Bezout): The degree of $Res(f,g)$ is at most 36.*

Factorization of the Resultant Polynomial

Conjecture 1. *Let $f = g = 0$ be our system of polynomial equations, i.e.*

$$\begin{aligned} f(\lambda,\mu) &= \boldsymbol{c}_B^T \overline{\boldsymbol{C}}_{\lambda,\mu}^T \boldsymbol{A} \overline{\boldsymbol{C}}_{\lambda,\mu} \boldsymbol{c}_B - 2|\boldsymbol{C}_{\lambda,\mu}| \boldsymbol{a}^T \overline{\boldsymbol{C}}_{\lambda,\mu} \boldsymbol{c}_B + a_0 |\boldsymbol{C}_{\lambda,\mu}|^2 = 0, \\ g(\lambda,\mu) &= \boldsymbol{c}_A^T \overline{\boldsymbol{C}}_{\lambda,\mu} \boldsymbol{B} \overline{\boldsymbol{C}}_{\lambda,\mu}^T \boldsymbol{c}_A - 2|\boldsymbol{C}_{\lambda,\mu}| \boldsymbol{b}^T \overline{\boldsymbol{C}}_{\lambda,\mu}^T \boldsymbol{c}_A + b_0 |\boldsymbol{C}_{\lambda,\mu}|^2 = 0, \end{aligned}$$

and the system $\boldsymbol{h}$ be defined as follows:

$$\boldsymbol{h}(\lambda,\mu) := (h_1, h_2, h_3)^T = \overline{\boldsymbol{C}}_{\lambda,\mu} \boldsymbol{c}_B - \overline{\boldsymbol{C}}_{\lambda,\mu}^T \boldsymbol{c}_A = \boldsymbol{0}.$$

Then the common roots of the polynomials $r_{ij} := Res(h_i, h_j)$, $1 \le i < j \le 3$, define a polynomial p that divides $Res(f,g)$.

Remark: Sufficient to solve p and $Res(f,g)/p$ of degree ≤ 24.

Tangential Intersection Points

Observation 1. *The tangential intersection points between Q_1 and Q_2 do fullfill the* LAGRANGE *conditions* $(i), \ldots, (iv)$*.*

We conject that that they can be determined by setting $\boldsymbol{x} = \boldsymbol{y}$, i.e. by solving the following bivariate system:

$$
\begin{aligned}
\boldsymbol{h}(\lambda,\mu) &= \overline{\boldsymbol{C}}_{\lambda,\mu}\boldsymbol{c}_B - \overline{\boldsymbol{C}}^T_{\lambda,\mu}\boldsymbol{c}_A \\
&= (|\boldsymbol{B}|\boldsymbol{a} - \boldsymbol{A}\overline{\boldsymbol{B}}\boldsymbol{b})\lambda^2 + (\boldsymbol{B}\overline{\boldsymbol{A}}\boldsymbol{a} - |\boldsymbol{A}|\boldsymbol{b})\mu^2 + \\
&\quad (|\boldsymbol{B}|\boldsymbol{T}_A\boldsymbol{a} - \boldsymbol{T}_{\overline{A}}\overline{\boldsymbol{B}}\boldsymbol{b})\lambda + (\boldsymbol{T}_{\overline{B}}\overline{\boldsymbol{A}}\boldsymbol{a} - |\boldsymbol{A}|\boldsymbol{T}_B\boldsymbol{b})\mu + \\
&\quad (\boldsymbol{T}_{A\overline{B}}\boldsymbol{a} - \boldsymbol{T}_{B\overline{A}}\boldsymbol{b})\lambda\mu + |\boldsymbol{B}|\overline{\boldsymbol{A}}\boldsymbol{a} - |\boldsymbol{A}|\overline{\boldsymbol{B}}\boldsymbol{b}.
\end{aligned}
$$

Summary: Surface-Surface Case

	Central Surfaces		Non-Central Surfaces		
	$a_0 \neq 0$	$a_0 = 0$	$\boldsymbol{a} \neq \boldsymbol{0}$	$\boldsymbol{a} = \boldsymbol{0}$	rg$\boldsymbol{A} = 1$
$a_0 \neq 0$	24	12	18	12	8
$a_0 = 0$		4	8	4	2
$\boldsymbol{a} \neq \boldsymbol{0}$			13	8	5
$\boldsymbol{a} = \boldsymbol{0}$				4	2

The Point-Curve Case

W.l.o.g. we can assume that the conic Q is embedded on the x_1-x_2-plane and centered around the origin, i.e.

$$Q: \quad \boldsymbol{q}(t) = r(t)\boldsymbol{u} + s(t)\boldsymbol{v}, \quad \boldsymbol{u}^T\boldsymbol{v} = 0.$$

Projecting the query point $\boldsymbol{p}$ onto the same plane yields a 2-D problem:

$$\min_t (\overline{\boldsymbol{p}} - r(t)\boldsymbol{u} - s(t)\boldsymbol{v})^2.$$

Setting the derivative of the distance function equal to zero, gives

$$f(t) \;=\; rr'\boldsymbol{u}^2 + ss'\boldsymbol{v}^2 - r'\overline{\boldsymbol{p}}^T\boldsymbol{u} - s'\overline{\boldsymbol{p}}^T\boldsymbol{v} = 0,$$

with $r' \equiv \frac{d\,r}{d\,t}$ and $s' \equiv \frac{d\,s}{d\,t}$.

The Curve-Curve Case

Given two conics P and Q, i.e.

$$\begin{aligned} P: \quad \boldsymbol{p}(t) &= r_1(t_1)\boldsymbol{u}_1 + s_1(t_1)\boldsymbol{v}_1, & \boldsymbol{u}_1^T\boldsymbol{v}_1 = 0, \\ Q: \quad \boldsymbol{q}(t) &= \boldsymbol{c}_2 + r_2(t_2)\boldsymbol{u}_2 + s_2(t_2)\boldsymbol{v}_2, & \boldsymbol{u}_2^T\boldsymbol{v}_2 = 0. \end{aligned}$$

The partial derivatives of $\delta^2(t_1, t_2) = (\boldsymbol{q}(t_2) - \boldsymbol{p}(t_1))^2$ yield the following system of bivariate equations:

$$\begin{aligned} f(t_1, t_2) &= [\boldsymbol{q}(t_2) - \boldsymbol{p}(t_1)]^T \left[-\frac{\partial r_1}{\partial t_1}\boldsymbol{u}_1 - \frac{\partial s_1}{\partial t_1}\boldsymbol{v_1} \right] = 0, \\ g(t_1, t_2) &= [\boldsymbol{q}(t_2) - \boldsymbol{p}(t_1)]^T \left[\frac{\partial r_2}{\partial t_2}\boldsymbol{u}_2 + \frac{\partial s_2}{\partial t_2}\boldsymbol{v_2} \right] = 0. \end{aligned}$$

Example: Distance Between Two Ellipses

Proposition 2. *The distance between two ellipses can be computed by solving polynomials of degree at most 16.*

Proof. If P and Q are both ellipses, we can write our conditions as:

$$\begin{aligned} f(t_1,t_2) &= (1+t_1^2)f_1(t_1,t_2) + (1+t_2^2)f_2(t_1) \\ &= (t_1+i)(t_1-i)f_1(t_1,t_2) + (t_2+i)(t_2-i)f_2(t_1), \\ g(t_1,t_2) &= (1+t_1^2)g_1(t_2) + (1+t_2^2)g_2(t_1,t_2) \\ &= (t_1+i)(t_1-i)g_1(t_2) + (t_2+i)(t_2-i)g_2(t_1,t_2), \end{aligned}$$

with polynomials f_i and g_i, $i = 1, 2$, of degrees at most 2 in t_1 and t_2. Since every $(\xi_1, \xi_2) \in \{-i, i\}^2$ solves the bivariate system, $(1+t_1^2)^2$ is a factor of $Res(f, g, t_2)$, whose degree is bounded by 20 (mixed-volume function). $\square$

Summary: Curve-Curve Case

	Ellipse	Hyberbola	Parabola	Line
Ellipse	16	16	12	4
Hyperbola		16	12	4
Parabola			9	3
Line				1

Natural Conics, Quadrics and the Torus

Natural Conics: Lines, Circles

Natural Quadrics: Planes, Spheres, Cylinders

Theorem 2. *The distance between two faces embedded on natural quadrics or the torus and trimmend by natural conics can be computed by solving univariate polynomials of degree at most 8.*